On interpolation of functions with a boundary layer by parabolic splines

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Abstract

A subject of the article is parabolic spline-interpolation of functions having high gradient domains. Uniform grids are proved inefficient. As for parabolic spline interpolation, asymptotically exact estimates on a class of functions with an exponential boundary layer are announced in regard with piecewise-uniform grids, concentrated in the boundary layer. There are obtained results showing non-uniform in small parameter estimates and divergence of interpolation processes. The author offered a modified parabolic spline for which uniform in small parameter interpolation error estimations were obtained. There are available results of numerical experiments confirming theoretical estimates.

Keywords: boundary layer; singular perturbation; parabolic spline-interpolation; error estimates

1. Introduction

Parabolic splines are widely applied to smooth interpolation of functions. Such splines are investigated in [1], [2] and in many other works. However, according to [3], [4], application of the polynomial spline-interpolation formulas to functions with large gradients in a boundary layer leads to essential errors of $O(1)$ type. In [4] there was constructed a non-polynomial analogue of a parabolic spline, which was exact for the boundary layer component. Numerical experiments have displayed an advantage in the accuracy of the constructed spline. However, the boundary layer component is not always known, thus, in such case, there is no reasonable alternative for condensing a grid in the boundary layer. This paper investigates a traditional parabolic spline interpolation in accordance with Subbotin's method [2] on the piecewise uniform grid condensing in the boundary layer. We obtained error estimates of interpolation that, however, are not uniform in small ε parameter. It is presented that an interpolation measure of inaccuracy of a boundary layer component might increase without limits while $\varepsilon \to 0$, thus, we need to develop special methods of interpolation for such type of tasks. Moreover, such an interpolation method is offered and investigated in our paper. We shall pass it noting that the divergence of interpolation processes by cubic and parabolic splines on nonuniform grids was regarded in papers [2], [5], [6] and some others. However, those examples of divergence provided there had an artificial origin or were implied with the help of Banach-Steinhaus theorem. In the present article, we have shown a divergence of functions describing solutions of a variety of applied problems. These results testify the need of development of universal high-order methods of smooth spline interpolation of functions on nonuniform grids and development of projective-grid methods of a high order for singular perturbed boundary value problems, because there is no need for application of the grid solution interpolation while using projective-grid methods.

2. Statement of the problem and main results

Let us introduce the following notations. Let $\Omega: 0 = x_0 < x_1 < \cdots < x_N = 1-$ partition of a segment $[0,1]$. Let $S(\Omega,k,1)$ be a space of polynomial splines [2] of defect 1 with k degree on grid Ω. We means that C, C_j are positive constants independent from ε and a number of grid nodes. We write $f = O(g)$ if $|f| \le C|g|$ and $f = O^*(g)$, whereas if $g = O(f)$, $C[a,b]$ is a space of continuous functions with a norm of $\|\cdot\|_{C[a,b]}$.

Let function $u(x)$ be as following:

$$u(x) = q(x) + \Phi(x),\ x \in [0,1], \tag{1}$$

$$\left|q^{(j)}(x)\right| \le C_1,\ \left|\Phi^{(j)}(x)\right| \le C_1 e^{-\alpha x/\varepsilon}/\varepsilon^j,\ 0 \le j \le 3. \tag{2}$$

Let us investigate a problem of parabolic spline-interpolation of the function (1).Let us introduce an auxiliary mesh:

$$\overline{\Omega} = \{\overline{x}_n, -1 \le n \le N\},\ \overline{x}_n = \frac{x_n + x_{n+1}}{2}, 0 \le n \le N-1,\ \overline{x}_{-1} = x_0 - \frac{h_1}{2}, \overline{x}_N = x_N + \frac{h_N}{2}.$$

First, let us look into the case of a uniform grid. Let N be a natural number, Δ is uniform grid with a step of $H = 1/N$ and nodes $x_n, n = 0,1,\cdots,N,$ partitioning of the interval is $[0,1]$. Let $g_2(x,u) \in S(\overline{\Omega},2,1)$ be an interpolation parabolic spline on the grid $\overline{\Omega}$, such that $g_2(x_n,u) = u(x_n),\ 0 \le n \le N,\ g_2{}'(0,u) = u'(0),\ g_2{}'(1,u) = u'(1).$ (3)

Let us define the uniform grid $h = 1/N$ at $[0,1]$ as Δ.

Theorem 1. *In case of a uniform grid $\Omega = \Delta$, there shall be such constant C for which the following estimate would be correct:*

$$\|u(x) - g_2(x,u)\|_{C[0,1]} \le C(N\varepsilon)^{-3}.$$

If in (1) $\Phi(x) = e^{-\alpha x/\varepsilon}$, then the following estimate also holds: $\|u(x) - g_2(x,u)\|_{C[0,1]} \ge C_1 \min\{(N\varepsilon)^{-1}, (N\varepsilon)^{-3}\}$.

Next, according to [7] let us define the grid Ω with nodes $x_n, n = 0,1,\cdots,N$, and steps $h_n = h = \frac{\sigma}{N/2}$, $n = 1,\cdots,\frac{N}{2}$, $h_n = H = \frac{1-\sigma}{N/2}, n = \frac{N}{2}+1,\cdots,N.$

In accordance with [7] let us define $\sigma = \min\left\{\frac{1}{2}, \frac{3\varepsilon}{\alpha}\ln N\right\}$. (4)

Due to [2] for interpolation parabolic spline $g_2(x,u) \in S(\overline{\Omega},2,1)$ the following error estimate is valid:

$$|g_2(x,u) - u(x)| \le C\|u^{(3)}\|_{C[0,1]} \max_n h_n^3. \quad (5)$$

Note that $g_2(x,u) = g_2(x,q) + g_2(x,\Phi)$, and due to (2),(5)

$$\|q(x) - g_2(x,q)\|_{C[0,1]} \le C_2 \max_n h_n^3 \le C_2 N^{-3}.$$

Henceforth, a spline interpolation approximating $u(x)$ with order $O(N^{-3}\ln^3 N)$, might be builtif and only if:

$$\|\Phi(x) - g_2(x,\Phi)\|_{C[0,1]} \le C_2 N^{-3}\ln^3 N. \quad (6)$$

In case when in (6) $\sigma = 1/2$, equality (6) shall be valid in account of Theorem 1 and due to the relation $N\varepsilon = O^*(N/\ln N)$. Thus, as following we assume that $\sigma < 1/2$. Yet, to keep it short we assign $g_2(x) = g_2(x,\Phi)$, $g_2(x) \in S(\overline{\Omega},2,1)$.

Theorem 2. *There are such constants C_2, C_3, which satisfy (6) if $N^{-1} \le C_3\varepsilon$.*

Theorem 3. *There are such constants C_4, C_5 and $\beta > 0$, independent from ε, N, that if $\varepsilon \le C_4 N^{-1}$, then*

$$\|g_2(x) - \Phi(x)\|_{C[x_n,x_{n+1}]} \le C_5 \begin{cases} N^{-3}\ln^3 N, 0 \le n \le N/2 - 1 \\ \dfrac{N^{-4}}{\varepsilon} e^{-\beta(n-N/2)}, N/2 \le n \le N-1 \end{cases}. \quad (7)$$

Next theorem shows us that estimates in (7) cannot be improved.

Theorem 4. Let $\Phi(x) = e^{-x/\varepsilon}$. *There are such constants $C_4, C_6, \beta_1 > 0$, independent from ε, N, that if $\varepsilon \le C_4 N^{-1}$, then*

$$\|g_2(x) - \Phi(x)\|_{C[x_n,x_{n+1}]} \ge C_6 \frac{N^{-4}}{\varepsilon} e^{-\beta_1(n-N/2)}, \qquad \frac{N}{2} \le n \le N-1.$$

Now let us construct a modified interpolation spline. Let $\tilde{x}_{N/2} = \overline{x}_{N/2} = (x_{N/2} + x_{N/2+1})/2$, $\tilde{x}_n = x_n$, $n \in [0, N/2-1] \cup [N/2+1, N]$. Let $gm_2(x,u)$ be the interpolation parabolic spline defined by the following conditions: $gm_2(\tilde{x}_n, u) = u(\tilde{x}_n)$, $n \in [0,N]$, $gm_2'(0,u) = u'(0)$, $gm_2'(1,u) = u'(1)$. The only difference between $gm_2(x,u)$ and

$g_2(x,u)$ is that the interpolation node $x_{N/2}$ is set as $\overline{x}_{N/2}$. The spline nodes whereas are not subject to any changes and coincide with the Ω nodes.

Theorem 5. *There are such independent from the* ε, N *constants, namely* $\gamma_0 > 0, C,$ *that if* $\varepsilon \ln N \le \gamma_0$, *then*

$$\|u(x) - gm_2(x,u)\|_{C[0,1]} \le CN^{-3} \ln^3 N. \tag{8}$$

Note 1. *Condition* $\varepsilon \ln N \le \gamma_0$ *is met when* $\varepsilon \le CN^{-1}$. *Thus in accordance with theorems 2,5 if we apply interpolation spline* $gm_2(x,u)$ *where* $\varepsilon = O(N^{-1})$ *and interpolation spline* $g_2(x,u)$ *where* $N^{-1} = O(\varepsilon)$ *,we obtain uniform in* ε, N *estimates (5),(8).*

3. Results of numerical experiments

Let us define the following function $u(x) = \cos\frac{\pi x}{2} + e^{-\frac{x}{\varepsilon}}, x \in [0,1]$. (9)

Results of calculations are provided in the three following tables. Given in the tables below are the maximum error evaluations of spline interpolation, calculated at nodes of the condensed grid, which in its turn is obtained from the initial mesh by splitting every single mesh interval into 10 equal parts. Table 1 contains interpolation error evaluations for the traditional parabolic spline on the uniform grid. Results confirm estimates of the Theorem 1 and an insufficiency of uniform grid for small values of ε. Table 2 provides errors of traditional parabolic spline in Shishkin meshes. It follows from the tables that measure of inaccuracy increases when ε decreases and N is fixed. Results of Table 3 for the modified spline, in contrary, refer to uniform convergence, thus, theoretical conclusions are confirmed.

Table 1. Errors of parabolic spline on the uniform grid

ε \ N	16	32	64	128	256	512
1	2.82e-7	1.76e-8	1.16e-9	1.02e-10	4.30e-11	2.68e-13
10e-1	3.43e-4	2.33e-5	1.51e-6	9.58e-8	6.03e-9	4.11e-10
10e-2	0.43	8.38e-2	9.72e-2	8.00e-4	5.59e-5	3.65e-6
10e-3	9.88	4.58	1.93	0.66	0.15	2.03e-2
10e-4	1.05e+2	5.23e+1	2.58e+1	1.25e+1	5.90	2.59
10e-5	1.06e+3	5.23e+2	2.64e+2	1.32e+2	6.56e+1	3.24e+1
10e-6	1.06e+4	5.30e+3	2.65e+3	1.33e+3	6.62e+2	3.30e+2
10e-7	1.06e+5	5.30e+4	2.65e+4	1.33e+4	6.63e+3	3.30e+3
10e-8	1.06e+6	5.30e+5	2.65e+5	1.33e+5	6.63e+4	3.31e+4

Table 2. Errors of parabolic spline on piecewise-uniform grid

ε \ N	16	32	64	128	256	512
1	9.38e-6	1.18e-6	1.47e-7	1.84e-8	2.31e-9	2.89e-10
10e-1	1.44e-3	2.50e-4	3.64e-5	4.90e-6	6.35e-7	8.09e-8
10e-2	4.37e-3	1.58e-3	4.49e-4	1.04e-4	2.15e-5	4.03e-6
10e-3	7.05e-3	1.58e-3	4.49e-4	1.04e-4	2.15e-5	4.03e-6
10e-4	7.32e-2	4.08e-3	4.49e-4	1.04e-4	2.15e-5	4.03e-6
10e-5	7.35e-1	4.11e-2	2.36e-3	1.39e-4	2.15e-5	4.03e-6
10e-6	7.35	4.11e-1	2.37e-2	1.40e-3	8.46e-5	5.18e-6
10e-7	73.5	4.11	2.37e-1	1.40e-2	8.46e-4	5.19e-5
10e-8	735	41.1	2.37	1.40e-1	8.46e-3	5.19e-4

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Table 3. Errors of modified parabolic spline on piecewise-uniform grid

N / ε	16	32	64	128	256	512
1	9.38e-6	1.18e-6	1.47e-7	1.84e-8	2.31e-9	2.89e-10
10e-1	1.44e-3	2.50e-4	3.64e-5	4.90e-6	6.35e-7	8.09e-8
10e-2	4.37e-3	1.58e-3	4.49e-4	1.04e-4	2.15e-5	4.03e-6
10e-3	4.37e-3	1.58e-3	4.49e-4	1.04e-4	2.15e-5	4.03e-6
10e-4	4.37e-3	1.58e-3	4.49e-4	1.04e-4	2.15e-5	4.03e-6
10e-5	4.37e-3	1.58e-3	4.49e-4	1.04e-4	2.15e-5	4.03e-6
10e-6	4.37e-3	1.58e-3	4.49e-4	1.04e-4	2.15e-5	4.03e-6
10e-7	4.37e-3	1.58e-3	4.49e-4	1.04e-4	2.15e-5	4.03e-6
10e-8	4.37e-3	1.58e-3	4.49e-4	1.04e-4	2.15e-5	4.03e-6